A spectral solution for linear diffusion in a simple landscape evolution model

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Introduction

- Spectral methods are powerful tools for solving elliptical and parabolic partial differential equations, widely used in other disciplines, though they appear rarely in landscape evolution modeling (Canuto 1987)
- Implicit and explicit spectral solutions are considered for solving the linear diffusion term of a simple 2D, loosely coupled landscape evolution model of elevation $z(x, y)$:

$$\frac{\partial z}{\partial t} = \nu - E + D\nabla^2 z$$

  Where t is time, ν is the uplift rate, E is the fluvial incision rate, and D is the hillslope diffusion constant.
- An implicit spectral solution for linear diffusion does not face the same stability constraints as a standard explicit finite difference formulation
- The simplest spectral solution requires periodic boundary conditions in both dimensions, which can create tessellating landscapes

Methods

The hillslope diffusion equation is:

$$\frac{\partial z}{\partial t} = D\nabla^2 z = D\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \quad (1)$$

By taking the Fourier transform of equation (1) in both x and y directions, we obtain the spectral form of this problem for wavenumbers k_x and k_y:

$$\frac{\partial \hat{\eta}_{k_x,k_y}}{\partial t} = D\big(-k_x^2\hat{\eta}_{k_x,k_y} - k_y^2\hat{\eta}_{k_x,k_y}\big) \quad (2)$$

where the wavenumber k_x is:

$$k_x = \frac{2\pi n_x}{N_x \Delta x}, \qquad n_x = [-N_x/2, N_x/2] \quad (3)$$

where N_x is the dimension of the grid in the x direction.

The **explicit form** of this equation considers the right hand side of the equation only at time step n:

$$\frac{\hat{\eta}^{n+1}_{k_x,k_y} - \hat{\eta}^{n}_{k_x,k_y}}{\Delta t} = D\big(-k_x^2\hat{\eta}^n_{k_x,k_y} - k_y^2\hat{\eta}^n_{k_x,k_y}\big) \quad (4)$$

The **implicit (Crank-Nicholson) form** of this equation uses both time n and time $n+1$ on the right side, and does not have the stability constraint of the explicit form:

$$\frac{\hat{\eta}^{n+1}_{k_x,k_y} - \hat{\eta}^{n}_{k_x,k_y}}{\Delta t} = \frac{D}{2}\big(-k_x^2\hat{\eta}^n_{k_x,k_y} - k_y^2\hat{\eta}^n_{k_x,k_y} - k_x^2\hat{\eta}^{n+1}_{k_x,k_y} - k_y^2\hat{\eta}^{n+1}_{k_x,k_y}\big) \quad (5)$$

For comparison, a simple explicit method often used to solve the diffusion equation is:

$$\frac{\eta^{n+1}_{i,j} - \eta^{n}_{i,j}}{\Delta t} = D\left(\frac{\eta^n_{i-1,j} - 2\eta^n_{i,j} + \eta^n_{i+1,j}}{\Delta x^2} + \frac{\eta^n_{i,j-1} - 2\eta^n_{i,j} + \eta^n_{i,j+1}}{\Delta y^2}\right) \quad (6)$$

Boundary Conditions

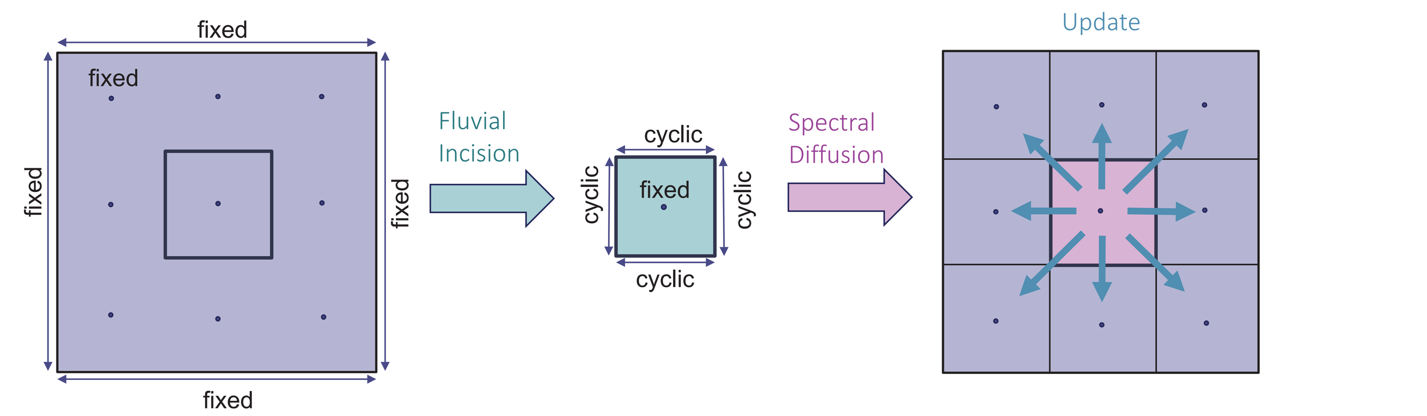


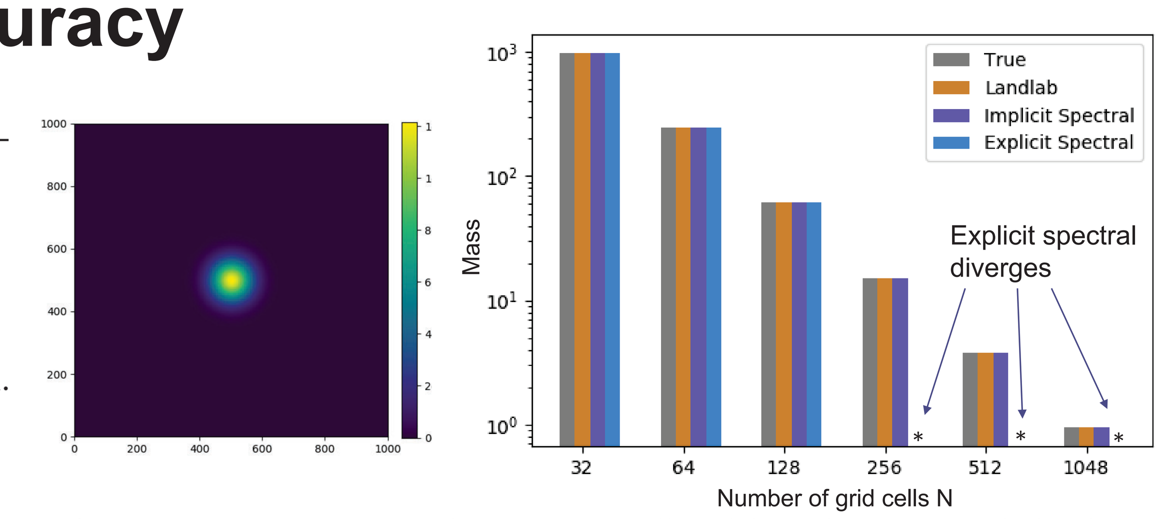
Figure 1. Schematic of boundary conditions and update configuration. This scheme is necessary to accommodate the lack of built-in periodic boundary conditions for fluvial incision in Landlab. The fluvial incision term is applied to the whole domain with fixed outer boundary conditions and fixed interior points. The center subdomain is extracted and the spectral diffusion method is applied. Each timestep, all subgrids are updated to match the center subgrid.

Analysis of stability and accuracy

When initial condition is a dirac-delta function, the exact solution to the diffusion equation is Gaussian:

$$z(x, y) = (1 * \Delta x \Delta y)\frac{1}{4\pi D t}\exp\left[-\frac{x^2}{4Dt} - \frac{y^2}{4Dt}\right]$$

Where $(1 * \Delta x \Delta y)$ is the initial mass of the delta function. Results are shown in Figures 2 and 3.



Figures 2,3. Test case, D = 1E-3, dt=2500, T=1E5. All models conform to true solution for the grid sizes tested, except the explicit spectral solution. At N=256, D*dt/dx² = 0.164

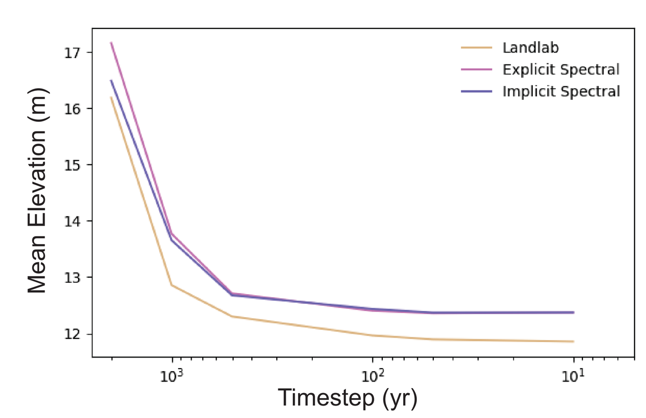


Figure 4. Mean elevation of the model domain for timesteps ranging from 2000 years to 10 years. Actual model realizations still differ even when mean elevation is the same. N=64, D = 1E-3 m^2/yr, k=3E-4 yr^{-1}, T=100 kyr

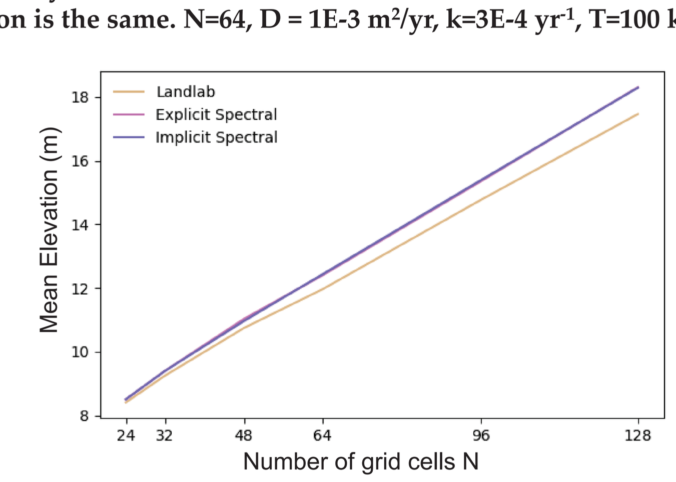


Figure 5. Mean elevation of the model domain for number of cells in the subgrid ranging from 24x24 to 128x128. Full results for selected N values are shown in Figure 6. dt=100, D = 1E-3 m^2/yr, k=3E-4 yr^{-1}, T=100 kyr.

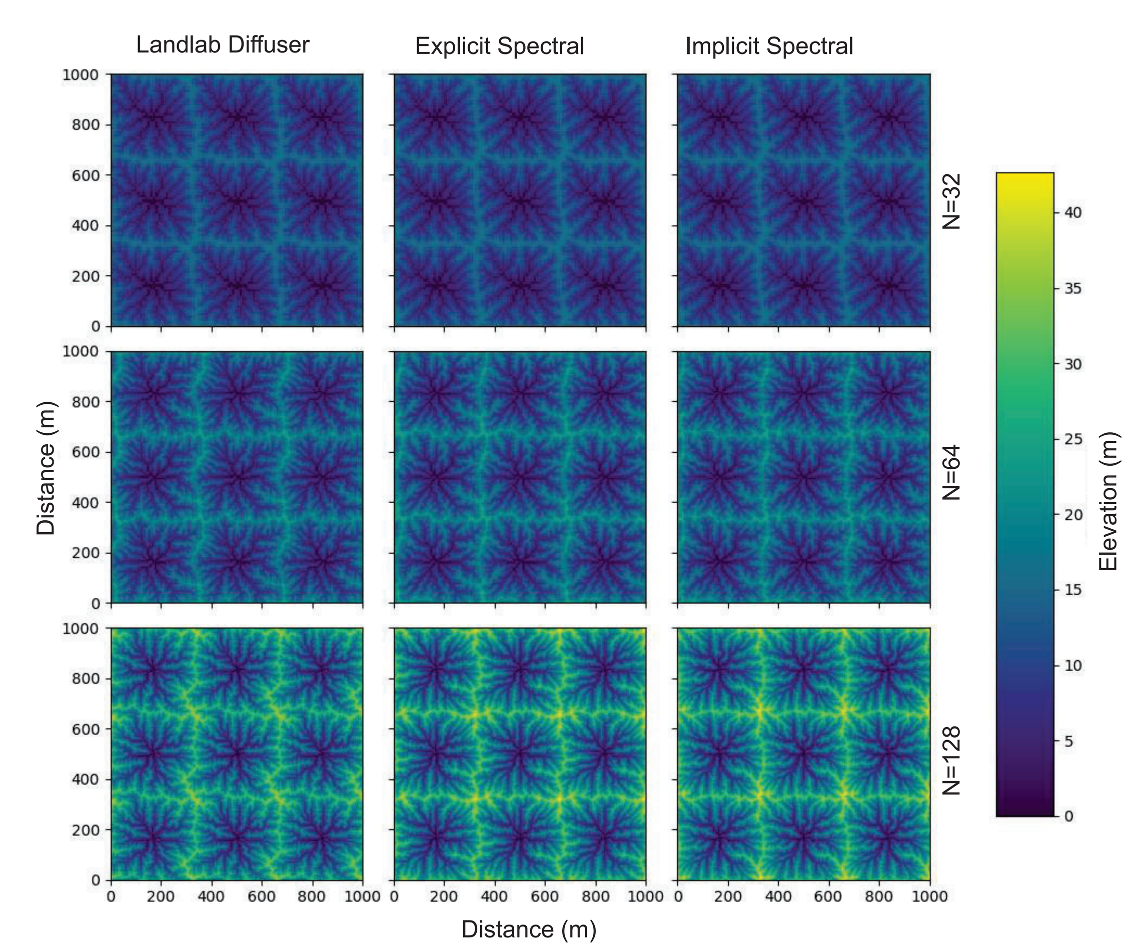


Figure 6. Comparison of the Landlab linear diffuser and spectral diffusers. The subgrid has periodic boundary conditions, and a single sink in the center of the subgrid. D=1E-3 m^2/yr, K=3E-4 yr^{-1}, m=0.5, n=1, uplift rate = 1E-3 m/yr, dt = 100 yr, T = 100kyr

Acknowledgments

Thanks to Dr. Katy Barnhart and Dr. Greg Tucker for productive discussions on Landlab and landscape evolution modelling.

References

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Discussion

- Method developed shows that spectral solutions are capable of performing linear diffusion in the context of landscape evolution modeling
- Solution has spectral accuracy and efficiency due to use of fast Fourier transforms
- Boundary condition differences and interaction between the fluvial incision and diffusion terms appear to cause differences in steady state landscapes when solved with different diffusion numerical methods
- Spectral solutions can be extended for non-periodic boundary conditions
- Landlab component could be developed, allowing for wider use of this tool.

Application: Drainage basins on a flat torus

Steady state drainage divides form Voronoi polygons

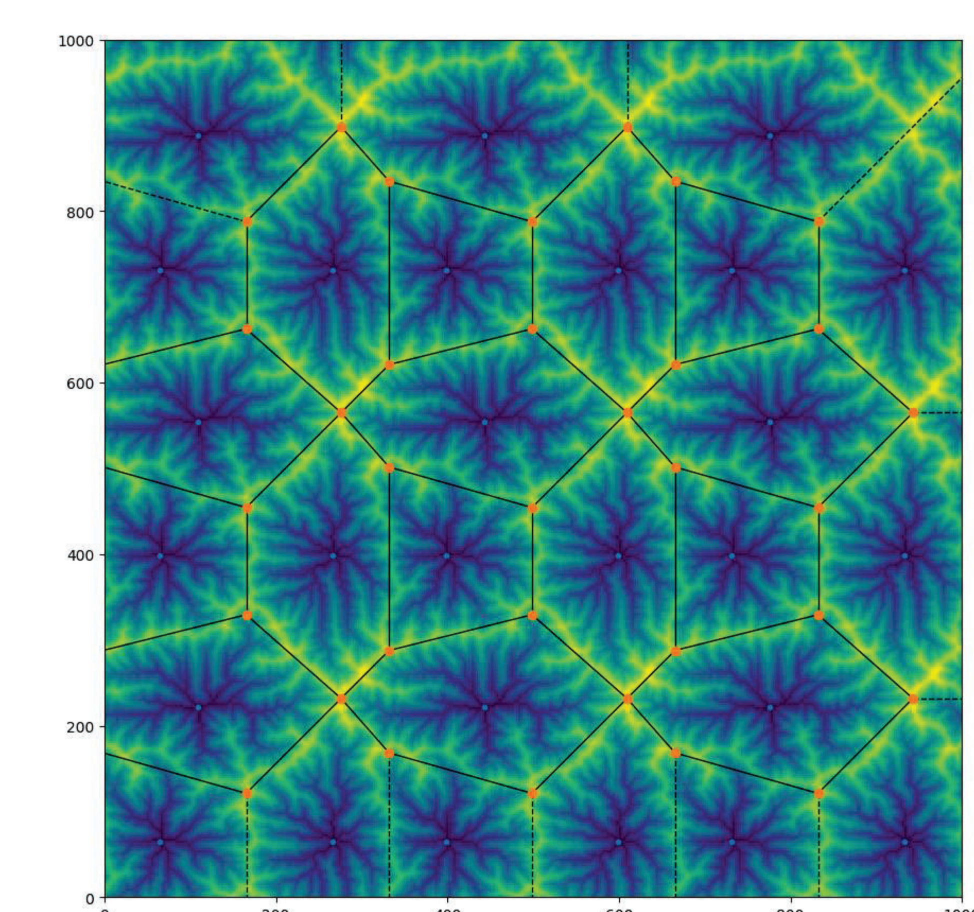

Figure 7. Results using implicit spectral diffuser and the same setup as in Figure 4, only three sinks are placed in each subgrid rather than one. For comparison, Voronoi polygons are overlain with the sinks as center points.

- The combination of point sinks and periodic boundary conditions illuminates optimality conditions embedded in this simple landscape evolution model
- In homogenous incision case with uniform baselevel, fluvial incision term in adjacent basins reaches steady state when drainage divides are equidistant from baselevel points (e.g. Willett et al 2014)